For suppose that $g(\mathrm{p}) = i$. Then $f(\mathrm{p}, i) = 0$, and for any $j < i, f(\mathrm{p}, j) \neq 0$. Since A represents f in Q, we have

$$\vdash_Q A(\mathbf{p}, \mathbf{i}, \mathbf{0}), \text{ and (if } i > 0), \tag{i}$$

$$\vdash_Q -A(\mathbf{p}, \mathbf{0}, \mathbf{0}), \tag{0}$$

$$\vdots$$

$$\vdash_Q -A(\mathbf{p}, \mathbf{i-1}, \mathbf{0}). \tag{$i-1$}$$

(0), ..., $(i-1)$, and 14.11 entail that

$$\vdash_Q \forall w(w < \mathbf{i} \rightarrow -A(\mathbf{p}, w, \mathbf{0})), \tag{$i+1$}$$

which, together with (i), entails that $\vdash_Q B(\mathbf{p}, \mathbf{i})$.

We must show that $\vdash_Q \forall x_{n+1}(B(\mathbf{p}, x_{n+1}) \rightarrow x_{n+1} = \mathbf{i})$. Assume $B(\mathbf{p}, x_{n+1})$, i.e., $A(\mathbf{p}, x_{n+1}, \mathbf{0})$ & $\forall w(w < x_{n+1} \rightarrow -A(\mathbf{p}, w, \mathbf{0}))$. From ($i$) and $\forall w(w < x_{n+1} \rightarrow -A(\mathbf{p}, w, \mathbf{0}))$, we have $-\mathbf{i} < x_{n+1}$. From $A(\mathbf{p}, x_{n+1}, \mathbf{0})$ and $(i+1)$, we have $-x_{n+1} < \mathbf{i}$. Thus by 14.13 we have $x_{n+1} = \mathbf{i}$. So $\vdash_Q \forall x_{n+1}(B(\mathbf{p}, x_{n+1}) \rightarrow x_{n+1} = \mathbf{i})$.

Exercises

14.1 Verify the following assertion: all recursive functions are representable in the theory ('R') whose language is L and whose theorems are the consequences in L of the following infinitely many sentences:

$\mathbf{i} \neq \mathbf{j}$ for all i, j such that $i \neq j$;

$\mathbf{i}+\mathbf{j} = \mathbf{k}$ for all i, j, k such that $i+j = k$;

$\mathbf{i} \cdot \mathbf{j} = \mathbf{k}$ for all i, j, k such that $i \cdot j = k$;

$\forall x(x < \mathbf{i} \rightarrow x = \mathbf{0} \vee \ldots \vee x = \mathbf{i-1})$ for all i;

and $\forall x(x < \mathbf{i} \vee x = \mathbf{i} \vee \mathbf{i} < x)$, for all i.

14.2 Show that none of the following sentences are theorems of Q:

(a) $\forall x\, x \neq x'$,

(b) $\forall x \forall y \forall z\, x+(y+z) = (x+y)+z$,

(c) $\forall x \forall y\, x+y = y+x$,

(d) $\forall x\, \mathbf{0}+x = x$,

(e) $\forall x\, x < x'$,

(f) $\forall x \forall y -(x < y \,\&\, y < x)$,

(g) $\forall x \forall y \forall z\, x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

(h) $\forall x \forall y\, x \cdot y = y \cdot x$,

(i) $\forall x\, \mathbf{0} \cdot x = \mathbf{0}$,

(j) $\forall x \forall y \forall z\, x \cdot (y+z) = x \cdot y + x \cdot z$.

Hint: Let a and b be two objects that are not natural numbers, and consider the following successor, addition, and multiplication tables:

x	x'
i	i'
a	a
b	b

$+$	j	a	b
i	$i+j$	b	a
a	a	b	a
b	b	b	a

$\cdot$	0	$j \neq 0$	a	b
0	0	0	a	b
$i \neq 0$	0	$i \cdot j$	a	b
a	0	b	b	b
b	0	a	a	a